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Solutions of Generalized Optimization Problems

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1. INTRODUCTION

Consider the following optimization problem: Let T be the closed interval $[t^0, t^1]$ of the real line, V an open set in n-dimensional Euclidean space E^n, B_0, and B_1 closed subsets of V. Let $\mathscr{F} = \{f(t, x)\}$ be a set of functions defined for (t, x) in $T \times V$ with range in E^n. It is desired to find an absolutely continuous function $x(t)$ for which

$$\dot{x}(t) = f(t, x(t)) \quad \text{almost everywhere,} \quad f \in \mathscr{F},\ (t, x(t)) \in T \times V, \tag{1.1}$$

$$x(t^0) \in B_0,\ x(t^1) \in B_1, \tag{1.2}$$

$$\phi(x(t^1)) = \min. \tag{1.3}$$

where ϕ is lower-semicontinuous on B_1. It will be assumed that all solutions[1] of (1.1) with $x(t^0)$ in B_0 are contained in V for t in T.

This problem is a generalization of the usual optimal control problem as shown in Section 3. Solutions of (1.1) and (1.2) will be called *solutions of a generalized optimization problem* or in short *generalized solutions.*

In Section 2, properties of generalized solutions are developed. It is shown that under suitable hypotheses, the set of generalized solutions is closed with respect to uniform convergence. As a simple consequence, the existence of a minimizing solution is established.

For applications in Section 3, it is convenient to associate with the set $\mathscr{F}$ an index set I which may be quite general. Whenever $\{i\}$ is a countable subset of I we will freely substitute for this set ordered by the integers, the integers themselves.

In Section 3, several particular cases resulting from specifying the sets I and $\mathscr{F}$ are considered. Choices that extend the theorems in Section 2 to the variable time problem, give relaxed solutions in the sense of Warga [*2*], and that give existence theorems for optimal control problems such as those in [*3*]–[*5*] are carried out in detail. Other possible choices are also indicated.

[1] The set of solutions may be empty as noted in the hypotheses to Corollary 2. For assumptions that this set be nonempty, see [*1*].

First-order necessary conditions for a generalized solution to be minimizing are presented in the paper immediately following in this issue.

2. Existence and Approximation of Generalized Solutions

Assume that there exists a function $k(t)$ integrable on T with L_1 norm K such that, for every i in an index set I for $\mathscr{F}$,

$$f_i(t, x) \text{ is Lebesgue-integrable on } T \text{ for fixed } x \text{ in } V, \tag{2.1a}$$

$$|f_i(t, x) - f_i(t, y)| \leqslant k(t)|x - y|, \qquad (t, x), (t, y) \in T \times V, \tag{2.1b}$$

$$|f_i(t, x)| \leqslant k(t) \qquad (t, x) \in T \times V. \tag{2.1c}$$

We omit the qualification "almost everywhere" here and in the following since it will be clear where it applies.

A sequence $f_n(t, x)$ of $\mathscr{F}$ will be called a *weak Cauchy sequence* if for some fixed x in V for any measurable subset E of T,

$$\lim_{m,n\to\infty} \int_E [f_n(t, x) - f_m(t, x)] = 0.$$

The set $\mathscr{F}$ will be said to be *weakly closed* if every weak Cauchy sequence of $\mathscr{F}$ is a weak convergent sequence. The weak Cauchy sequence is a *weak convergent sequence* if there is a function $f(t, x)$ in $\mathscr{F}$ to which $f_n(t, x)$ converges. The set $\mathscr{F}$ will be said to be *weakly sequentially compact* if for fixed x, any sequence $f_n(t, x)$ contains a weak Cauchy subsequence. As a consequence of (2.1c), $\mathscr{F}$ is weakly sequentially compact (see Theorem 2.2), but not necessarily weakly closed.

Lemma 2.1. *Every sequence $f_i(t, x)$ of $\mathscr{F}$ contains a subsequence $f_n(t, x)$ that is a weak Cauchy sequence for each x in V.*

Let V_0 be a countable dense set of V and let x_1 be a point of V_0. Then since $\mathscr{F}$ is weakly sequentially compact, the sequence $f_i(t, x_1)$ contains a weak Cauchy subsequence $f_n(t, x_1)$. To repeat, for each point x_k in V_0, a subsequence [again call it $f_n(t, x)$] can be extracted which is weakly Cauchy for every x in V_0. Let y be any point of V and E any measurable subset of T. Then

$$\begin{aligned}\left|\int_E [f_n(t, y) - f_m(t, y)]\right| &\leqslant \left|\int_E [f_n(t, y) - f_n(t, x)]\right| + \left|\int_E f_n(t, x) - f_m(t, x)]\right| \\ &\quad + \left|\int_E [f_m(t, x) - f_m(t, y)]\right| \\ &\leqslant 2|x - y \int_E k(t) + \left|\int_E [f_n(t, x) - f_m(t, x)]\right|\end{aligned}$$

using (2.1b). The first term of the right-hand side can be made arbitrarily small be choice of x in V_0 and the second term goes to zero as m, n tend to infinity for any such choice.

As a consequence of this lemma, it will not be necessary to specify x when extracting weak Cauchy sequences.

LEMMA 2.2. *If $x(t)$ is a continuous function from T to V and the sequence $f_n(t, x)$ in $\mathscr{F}$ is a weak Cauchy sequence, then $f_n(t, x(t))$ contains a weak Cauchy subsequence.*

By Lemma 2.1, the result is true for step functions. The conclusion follows from approximating $x(t)$ uniformly by a step function and using (2.1b).

LEMMA 2.3. *If $x_n(t)$ is a uniformly convergent sequence of continuous functions with limit $x(t)$ in V and $f_n(t, x)$ is a weak Cauchy sequence, then $f_n(t, x_n(t))$ contains a weak Cauchy subsequence.*

Let E be any measurable subset of T. Then $x(t)$ is continuous and

$$\begin{aligned}\left|\int_E [f_n(t, x_n(t)) - f_m(t, x_m(t))]\right| &\leqslant \left|\int_E [f_n(t, x_n(t)) - f_n(t, x(t))]\right| \\ &\quad + \left|\int_E [f_n(t, x(t)) - f_m(t, x(t))]\right| \\ &\quad + \left|\int_E [f_m(t, x(t)) - f_m(t, x_m(t))]\right| \\ &\leqslant \sup_t(|x_n(t) - x(t)| + |x_m(t) - x(t)|) \int_E k(s) \\ &\quad + \left|\int_E [f_n(t, x(t)) - f_m(t, x(t))]\right|.\end{aligned}$$

The limit of the first term is zero by hypothesis and, by Lemma 2.2, a subsequence of the second is a Cauchy sequence.

We remark that if the sequence $f_n(t, x)$ converges weakly to a function $f(t, x)$, a subsequence of $f_n(t, x_n(t))$ converges weakly to $f(t, x(t))$.

Now for absolutely continuous functions $x(t)$, let

$$\mathscr{X} = \{x(t) \mid x(t^0) \in B_0,\ \dot{x}(t) = f_i(t, x(t)),\ \text{some } i \in I,\ t \in T\}. \tag{2.2}$$

THEOREM 2.1. *The set $\mathscr{X}$ is closed with respect to uniform convergence if $\mathscr{F}$ weakly closed.*

Let $x_n(t)$ be a sequence in $\mathscr{X}$ which converges to $x(t)$. Then $x(t)$ is continuous and $x(t^0)$ is in the closed set B_0. Let $f_n(t, x)$ be the functions in $\mathscr{F}$ for which

$$\dot{x}_n(t) - f_n(t, x_n(t)), \quad t \in T.$$

Since the set of functions $f_n(t, x)$ is weakly sequentially compact and $\mathscr{F}$ is closed, there is a subsequence, again call it $f_n(t, x)$, that converges weakly to $f(t, x)$ in $\mathscr{F}$. Convergence can be made independent of x by Lemma 2.1. Thus,

$$\begin{aligned} x(t) &= \lim x_n(t) = \lim x_n(t^0) + \lim \int_{[t^0, t]} f_n(s, x_n(s)) \\ &= x(t^0) + \lim \int_{[t^0, t]} [f_n(s, x_n(s)) - f(s, x(s))] + \int_{[t^0, t]} f(s, x(s)), \end{aligned}$$

where the limit is zero for a subsequence, by Lemma 2.3.
Let

$$X = \{x(t^1) \mid x(t) \in X\}. \tag{2.4}$$

We will call X the *generalized attainable set.*

Corollary 1. *The set X is closed if $\mathscr{X}$ is closed and is bounded if B_0 is bounded.*

The first conclusion is obvious and the second follows from (2.1c) and the resulting inequality

$$| x(t^1) | \leqslant K(t^1 - t^0) + | x(t^0) |.$$

Now let

$$\mathscr{Y} = \{x(t) \in \mathscr{X} \mid x(t^1) \in B_1\} \tag{2.5}$$

and

$$Y = \{x(t^1) \mid x(t) \in \mathscr{Y}\}. \tag{2.6}$$

Elements of $\mathscr{Y}$ will be called *generalized solutions.* Note that as a consequence of Corollary 1, Y is closed if $\mathscr{X}$ is closed.

Corollary 2. *If either B_0 or B_1 is bounded, $\mathscr{X}$ is closed and Y is not empty, there exists a minimizing generalized solution.*

This follows from the lower-semicontinuity of $\phi(x(t^1))$ on B_1, Corollary 1 and the equivalent definition

$$Y = X \cap B_1 .$$

Assume $\mathscr{F}$ is not weakly closed and let $\mathscr{F}^*$ be its weak closure. Let $\mathscr{X}^*$ be the set of solutions corresponding to $\mathscr{F}^*$. Elements of $\mathscr{X}^*$ will be called *weak* generalized solutions. The following approximation is immediate.

Corollary 3. *Every weak generalized solution is the uniform limit of a sequence of generalized solutions.*

The next theorem is taken from [6] for reference.

THEOREM 2.2. *A subset $\mathscr{F}$ of L_1 is weakly sequentially compact iff it is bounded and the countable additivity of the integrals.*

$$\int_E f$$

is uniform with respect to f in $\mathscr{F}$.

Note that by Theorem 2.2, $\mathscr{F}$ is not weakly sequentially compact if (2.1c) is replaced by

$$\|f(t, x)\| \leqslant K.$$

The properties of the space $L_1(T)$ used in all proofs hold for the more general Lebesgue spaces $L_p(S, \Sigma, \mu, B)$, $p \geqslant 1$ where μ is a countably additive complex or extended real-valued function on a σ-field Σ of subsets of a set S, and the functions to be integrated with μ have their values in a Banach space B[6]. Also the conclusion of Lemma 2.3 can readily be shown to hold under much weaker hypotheses then continuity of $x_n(t)$, uniform convergence and (2.1b) provided integrability of the functions $f_n(t, x_n(t))$ is assured. This suggests that the proof of Theorem 2.1 can be carried out for choices of functional equations other than (1.1), since the only use of (1.1) was in the writing of $x(t)$ as the Lebesgue integral of its derivative.

3. PARTICULAR CHOICES OF $\mathscr{F}$

A. *Variable-Time Problem*

Let T_0 and T_1 be subsets of T and let $J = I \times T_0 \times T_1$ where I is an index set for a set of functions defined on $T \times V$. Then the variable-time problem is given by

$$(\alpha, x(\alpha)) \in T_0 \times B_0\,, \; (\beta, x(\beta)) \in T \times B_1 \tag{1.2'}$$

and

$$\mathscr{F}^* = \{f_j(t, x) \mid j \in J\},$$

where

$$f_j(t, x) = f_i(t, x), \qquad \sigma \leqslant t \leqslant \tau,\ x \in V,$$
$$= 0, \qquad \text{otherwise.}$$

When $x^1(t) = t$ and $g(x(t^1)) = x^1(t^1)$, the minimum-time problem results.

B. *Relaxed Solutions*

Let U be an arbitrary set. We will say that a measurable function $u(t)$ is *countably simple* if it takes on only countable values in U for t in T. Let

$$I = \{u(t) \mid u(t) \text{ is a countably simple function}\}. \tag{3.1}$$

Assume there exists a function $g(t, x, u)$ defined on $T \times V \times U$ such that

$$g(t, x, u(t)) \text{ is integrable on } T \text{ for fixed } x \text{ in } V \text{ and } u(t) \text{ in } I, \tag{3.2a}$$

$$| g(t, x, u) - g(t, y, u) | \leqslant k(t) | x - y |, \ (t, x, u), (t, y, u) \in T \times V \times U, \tag{3.2b}$$

$$| g(t, x, u) | \leqslant k(t) \qquad (t, x, u) \in T \times V \times U. \tag{3.2c}$$

Then let $\mathscr{F}$ be the weak closure of the set of functions given by

$$\mathscr{F}^* = \{f(t, x) = g(t, x, u(t)), \ u(t) \in I\}. \tag{3.3}$$

Condition (2.1) is satisfied for $\mathscr{F}$ as a consequence of (3.2).

THEOREM 3.1. *For $\mathscr{F}$ as defined, every generalized solution is a relaxed solution and conversely.*

If a sequence $f_n(t, x)$ in $\mathscr{F}$ converges weakly to $f(t, x)$, then for each (t, x) in $T \times V$

$$f(t, x) \subset \text{convex closure } \{f_n(t, x)\}.$$

Hence by definition (see [2]), every generalized solution of

$$\dot{x} = f(t, x) \qquad (t, x) \in T \times V$$

is a relaxed solution.

Conversely, as shown in [2], every relaxed solution is the uniform limit of a sequence of solutions whose controls are countably simple functions. Thus the sequence is in $\mathscr{X}$ and by Theorem 2.1 so is its limit.

As a consequence, Theorems 2.2 and 3.1 of [2] follow from our Theorem 2.1 and Corollary 2. Hypothesis (3.3.2) of [2] is not required for the latter result.

C. *Existence Theorems for Optimal Control*

Next assume that U is a closed set in E_m and let

$$I = \{u(t) \mid u(t) \text{ is a measurable function from } T \text{ to } U\}. \tag{3.4}$$

We suppose that there is a function $g(t, x, u)$ that, in addition to satisfying (3.2) for I given by (3.4), is continuous in u on U. We further suppose that for each (t, x) in $T \times V$, $g(t, x, U)$ is a closed convex set in E^n.

THEOREM 3.2. *With these additional hypotheses $\mathscr{X}$ is closed.*

Let $x_n(t)$ be a sequence in $\mathscr{X}$ converging uniformly to a function $x(t)$. We will show that $x(t)$ is in $\mathscr{X}$.

To $x_n(t)$ in $\mathscr{X}$, there corresponds a sequence in $\mathscr{F}$ of which a subsequence $f_n(t, x)$ is weakly Cauchy with limit $f(t, x)$. By Lemma 2.3, a subsequence of $f_n(t, x_n(t))$

converges weakly to $f(t, x(t))$. Since $g(t, x, U)$ is convex and closed for each (t, x) in $T \times V$,

$$f(t, x(t)) \subset g(t, x(t), U).$$

Hence there is a $u(t)$, not necessarily unique or measurable, such that

$$f(t, x(t)) = g(t, x(t), u(t)), \qquad t \in T.$$

The function $g(t, x(t), u)$ is integrable in t for each u in U. As a consequence, for each integer k there is a disjoint decomposition of T into measurable sets $T_{kj}(j = 1,..., q_k)$ and a point t_{kj} in each T_{kj} such that

$$\sup | g(t, x(t), u(t_{kj})) - g(t_{kj}, x(t_{kj}), u(t_{kj})) | \leqslant 1/2k, \qquad t \in T_{kj}$$

for each $j(j = 1,..., q_k)$. Then by the triangle inequality.

$$\sup | g(r, x(r), u(t_{kj})) - g(s, x(s), u(t_{kj})) | \leqslant 1/k, \qquad r, s \in T_{kj}$$

For each k let $u_k(t)$ be the step function given by

$$u_k(t) = u(t_{kj}), \qquad t \in T.$$

Then

$$|f(t, x(t)) - g(t, x(t), u_k(t)) | \leqslant 1k, \qquad t \in T.$$

Now

$$u_0(t) = \overline{\lim}\, u_k(t)$$

is in I since it is measurable on T and U is closed. From the continuity of g with respect to u,

$$f(t, x(t)) = g(t, x(t), u_0(t)), \qquad t \in T.$$

Hence $x(t)$ is a solution of

$$\dot{x} = g(t, x(t), u_0(t))$$

and is in $\mathscr{X}$, as to be shown.

Theorem 3.2 together with Corollary 2 of Theorem 2.1 give an existence theorem for optimal control problems which include the results in [3]–[5] in addition to the well known results for the linear problem. Note that when $g(t, x, u)$ is linear in u, the convexity of $g(t, x, U)$ is convexity of U.

D. Control Problems with Parameters

Let W be an arbitrary set and suppose that $g(t, x, u, w)$ is a function defined on $T \times V \times U \times W$. Then the control problem with parameters is given by

$$f_j(t, x) = g(t, x, u_i(t), w_k), j \in J, (t, x) \in T \times V$$

where $J = I \times W$.

E. Differential Games

If $I = J \times K$ where J and K are sets of strategies and (1.3) is replaced by

$$\min_J \max_K \phi(x(t^1)), \tag{1.3'}$$

where ϕ is continuous on B_1, the results of Section 2 hold for differential games given by (1.1).

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